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On k-pairable regular graphs

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ABSTRACT

Let k be a positive integer. A graph G is said to be *k-pairable* if its automorphism group contains an involution ϕ such that $d(x, \phi(x)) \geq k$ for any vertex x of G. The *pair length* of a graph G, denoted as $p(G)$, is the maximum k such that G is k-pairable; $p(G) = 0$ if G is not k-pairable for any positive integer k. Some new results have been obtained since these concepts were introduced by Chen [Z. Chen, On k-pairable graphs, Discrete Mathematics 287 (2004) 11–15].

In the present paper, we first introduce a new concept called *strongly induced cycle* and use it to give a condition for a graph G to have $p(G) = k$. Then we consider the class $\mathbb{G}(r, k)$ of prime graphs which are r-regular and have pair length k. For any integers $r, k \geq 2$, except $r = k = 2$, we show that the set $\mathbb{G}(r, k)$ is not empty, determine the minimum order of a graph in $\mathbb{G}(r, k)$, and give a construction for such a graph with the minimum order. With this approach, we also obtain the minimum order of an r-regular graph with pair length k for any integers $r, k \geq 2$. Finally, we post an open question for further research.

1. Introduction

All graphs considered here are finite, connected and simple. Motivated by an elegant result of Graham et al. [8] on spanning trees of a graph with an antipodal isomorphism, Chen [6] introduced in 2004 the concept of k-pairable graphs and extended the result in [8] to this larger class of graphs.

Definition 1.1 (*[5,6]*). Let k be a positive integer. A graph G is said to be *k-pairable* if its automorphism group contains an involution ϕ such that $d(x, \phi(x)) \geq k$ for any $x \in V(G)$.

Note that in this definition, an *involution* means a bijective map ϕ on $V(G)$ such that $\phi(\phi(x)) = x$ for all $x \in V(G)$.

By Definition 1.1, any k_1-pairable graph is k_2-pairable if integers $k_1 \geq k_2 \geq 1$. For brevity, we may call a k-pairable graph with $k \geq 1$ simply as a *pairable graph* when there is no confusion.

Clearly, any pairable graph has an even number of vertices. It was pointed out in [6] that such graphs have some special kind of symmetry which is different from the well known types of symmetry such as vertex-transitivity, edge-transitivity, or distance-transitivity. For example, a path P_{2n} ($n \geq 1$) with $2n$ vertices is 1-pairable, but it does not have those well known types of symmetry when $n > 1$. On the other hand, the Petersen graph possesses the above familiar types of symmetry [10], but it is not a pairable graph (see Example 3.1).

A special type of pairable graphs which are called *S-graphs* have been studied by many researchers (see [1,2,9] and the references therein). Che [5] studied another special class of pairable graphs which are called *uniquely k-pairable graphs*, and

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obtained a characterization of this class of graphs in terms of the prime factor decomposition with respect to Cartesian product.

To further study k-pairable graphs, Chen [6] defined a new graph parameter $p(G)$, called the *pair length* of a graph G. This parameter measures the maximum distance, in some sense, between a subgraph induced by half the vertices of G and its isomorphic images induced by the other half of $V(G)$.

Definition 1.2 (*[6]*)**.** The *pair length* of a graph G, denoted as $p(G)$, is the maximum k such that G is k-pairable; $p(G) = 0$ if G is not k-pairable for any positive integer k.

The pair lengths of cycles C_n, complete graphs K_n and complete bipartite graphs $K_{n,n}$ were given in [6]: $p(C_n) = \frac{n}{2}$ and $p(K_n) = 1$ if n is even, $p(C_n) = p(K_n) = 0$ if n is odd; and $p(K_{n,n}) = 2$ or 1 depending on n is even or odd. It was shown [6] that the pair length of a tree is either 0 or 1. A characterization of 1-pairable trees was given by Che [4]: A tree T has $p(T) = 1$ if and only if T has an edge xy such that there exists an isomorphism f between the two connected components of $T \setminus xy$ satisfying $f(x) = y$.

The study on the pair length of a Cartesian product graph was initiated by Chen [6]. Recall that the *Cartesian product* of two graphs G and H, denoted as $G \square H$, has the vertex set $V(G) \times V(H)$ and (g_1, h_1) adj (g_2, h_2) if either $g_1 = g_2$ in G and h_1 adj h_2 in H or g_1 adj g_2 in G and $h_1 = h_2$ in H. Chen showed that the pair length of a Cartesian product graph is not less than the sum of pair lengths of its factors and posted the question that if the equality is always true in general. This question was answered affirmatively by Christofides in [7]. Using this result, we can easily obtain the following proposition on regular pairable graphs.

Proposition 1.1. *Let $G = G_1 \square G_2$. Then G is an r-regular graph with pair length k ($r, k > 0$) if and only if both G_1 and G_2 are regular graphs with their degree sum equal to r and their pair length sum equal to k.*

Let G be an r-regular graph with pair length k. By the above result, we can see that $G \square C_n$ is $(r + 2)$-regular with pair length k if n is odd, with pair length $k + \frac{n}{2}$ if n is even; and that $G \square K_n$ is $(r + n - 1)$-regular with pair length k if n is odd, and with pair length $k + 1$ if n is even. Then it is not difficult to verify the existence of an r-regular graph with pair length k for any given integers $r, k \geq 2$. (Note that the cases when $r = 1$ or $k = 1$ are trivial.) Since prime graphs are building blocks of all graph structures, it is natural to ask: Does there exist a prime graph which is r-regular and has pair length k for any given integers $r, k \geq 2$?

This question is answered completely in the present paper. In fact, we have done much more than that. By introducing a new concept called a strongly induced cycle, we give a condition for a graph G to have $p(G) = k$. Then we determine the minimum order of an r-regular prime graph with pair length k, and construct such a graph with the minimum order for any given integers $r, k \geq 2$ (excluding $r = k = 2$). With this approach, we also obtain the minimum order of an r-regular graph with pair length k for any integers $r, k \geq 2$.

2. Preliminary

We follow the standard terminology. The vertex set of a graph G is denoted as $V(G)$; the number of vertices of G is called the *order* of G, and denoted by $|V(G)|$. For $x, y \in V(G)$, x adj y means that x is adjacent with y in G. The *distance* between x and y in G is denoted as $d_G(x, y)$ or simply as $d(x, y)$ if it causes no confusion.

A *matching* of a graph G is a set of disjoint edges of G. If a matching consists of m edges, then it is also called an *m-matching*. A *perfect matching* (or 1-*factor*) of a graph G is a matching of G that covers all vertices. A 1-*factorization* of a graph G is a collection of 1-factors such that every edge of G is in exactly one of these 1-factors. A graph G is said to be 1-*factorable* if it admits a 1-factorization. It is well known [12] that every complete graph K_{2n} ($n \geq 1$) is 1-factorable, and so it has $2n - 1$ pairwise disjoint perfect matchings (or 1-factors).

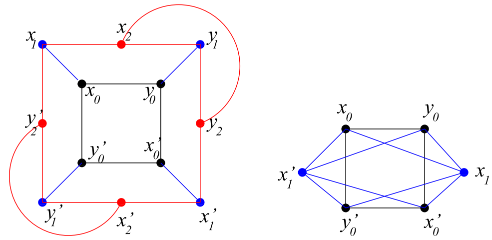
An *isomorphism* between two graphs G and H is a bijection f from $V(G)$ to $V(H)$ such that u adj v in G if and only if $f(u)$ adj $f(v)$ in H for any $u, v \in V(G)$. An *automorphism* of a graph G is an isomorphism from G onto itself. The set of all automorphisms of a graph G forms a group, which is called the *automorphism group* of G and denoted by $Aut(G)$.

It is clear that if a graph G is k-pairable, then its vertex set can be partitioned into disjoint pairs, that is, $V(G) = \bigcup_{x \in V(G)} \{x, \phi(x)\}$, where $\phi \in Aut(G)$ is an involution of $V(G)$. We call this partition of $V(G)$ a *k-pair partition* of G. For each vertex x of G, $\phi(x)$ is called the *mate* of x, and denoted as $x' = \phi(x)$. By definition, if x' is the *mate* of x, then x is the mate of x', and $d(x, x') \geq k$ for any $x \in V(G)$. Moreover, x adj y if and only if x' adj y' for any $x, y \in V(G)$. Therefore, a k-pairable graph has such a special symmetry that it can be obtained by appropriately adding edges to connect two disjoint isomorphic induced subgraphs, and the distance between any vertex and its image under the isomorphism is at least k.

The *eccentricity* of a vertex $u \in V(G)$ is defined as $e(u) = \max_{v \in V(G)} d(u, v)$. The *radius* of G is $r(G) = \min_{u \in V(G)} e(u)$. A vertex u of G with $e(u) = r(G)$ is called a *central vertex* of G. An upper bound of the pair length of a graph can be given by its radius.

Lemma 2.1 (*[6]*)**.** *For any graph G, $p(G) \leq \min\{r(G), \frac{|V(G)|}{2}\}$.*

The Cartesian product is a widely used graph operation which is associative and commutative. A Cartesian product graph $G = G_1 \square G_2$ is connected if and only if both factors G_1 and G_2 are connected. For any given vertex v in G_2, the subgraph of G



(a) G. (b) H.

Fig. 1. Prime graphs.

induced by the vertex set $\{(x, v) : x \in V(G_1)\}$ is isomorphic to G_1, and so it is called a G_1-*layer* of G. Clearly, the number of G_1-layers in G is $|V(G_2)|$. The G_2-layers of G are defined similarly. For a comprehensive introduction on graph products, the reader is referred to the book [11] by Imrich and Klavžar.

A graph is called *prime* if it cannot be represented as a Cartesian product of nontrivial graphs. In other words, G is prime if the equality $G = G_1 \square G_2$ implies that G_1 or G_2 is the one-vertex graph K_1. The well-known Sabidussi–Vizing Theorem [14] asserts that any connected graph G can be decomposed uniquely (up to order) into prime factors with respect to the Cartesian product.

To prove that a graph is prime, we have the following two useful lemmas.

Lemma 2.2. *If a graph G has an edge which is not contained in any induced 4-cycle of G, then G is prime.*

Proof. By contradiction. If G is not prime, then it can be represented as a Cartesian product of two nontrivial graphs. By the definition of a Cartesian product graph, every edge of G must be contained in an induced 4-cycle of G. This contradicts the assumption on G. $\square$

To give the next lemma, we first recall the following definition from [13]. A graph G is called *strongly triangulated* if any two vertices of G can be joined by a sequence of triangles such that any two consecutive ones have an edge in common.

Lemma 2.3. *Every strongly triangulated graph is prime.*

Proof. Let G be a strongly triangulated graph. Suppose that $G = G_1 \square G_2$. We only need to show that one of the two factors G_i $(i = 1, 2)$ must be K_1. By the definition of Cartesian product, any edge of G belongs to exactly one layer of G. Thus, each triangle of G must belong to exactly one layer of G, and so any two triangles with a common edge must be contained in exactly one layer of G. Then, since G is strongly triangulated, all vertices of G must be in exactly one layer, say, a G_1-layer. It implies that G_2 must be K_1, and so G is prime. $\square$

For example, the graph G in Fig. 1(a) has an edge x_2y_2 that is not contained in any induced 4-cycle of G. So G is prime by Lemma 2.2, although it is not strongly triangulated. On the other hand, each edge of the graph H in Fig. 1(b) is contained in an induced 4-cycle of H, so we can not apply Lemma 2.2 to show that H is prime. But H is strongly triangulated, so it is prime by Lemma 2.3.

3. Main results

In this section, we first introduce a new concept called a strongly induced cycle, and use the concept to give a condition for a graph to have pair length $k \geq 1$. Then we apply it to study a class of prime graphs which are regular and pairable. Let $r, k \geq 2$ be integers. Let $\mathbb{G}(r, k)$ denote the set of all prime graphs which are r-regular and have pair length k. We will determine the minimum order of a graph in $\mathbb{G}(r, k)$ and provide a construction of such a graph with the minimum order for $r, k \geq 2$ except $r = k = 2$. In fact, the obtained result also holds for a larger class of graphs: the set of r-regular graphs with pair length k (prime and non-prime).

Definition 3.1. An induced cycle C of a graph G is called a *strongly induced cycle* if $d_C(x, y) = d_G(x, y)$ for any two vertices x and y of C.

It is clear that if $n = 3, 4, 5$, then an induced n-cycle of G is just a strongly induced n-cycle of G, but it is not necessarily true when $n > 5$.

Theorem 3.1. *Let G be a simple graph with pair length $p(G)$. Then*

(i) *$p(G) = 1$ if and only if G is 1-pairable and for any 1-pair partition Π of G there is an edge e_Π of G such that the two end vertices of e_Π are a pair of mates.*
(ii) *$p(G) = k(> 1)$ if and only if G is k-pairable and for any k-pair partition Π of G there is a strongly induced $2k$-cycle C_Π of G such that the mate of any vertex on C_Π remains on C_Π.*

(Note: In general, the above edge e_Π and the strongly induced $2k$-cycle C_Π are not fixed, since they depend on the pair partition Π considered.)

Proof. (i) The necessity is obvious. So we only prove the sufficiency. Let G be a 1-pairable graph satisfying the given condition. Then $p(G) \geq 1$. Let Π be a $p(G)$-pair partition of G. Then Π is also a 1-pair partition of G. By the given condition, there is an edge e_Π of G such that the two end vertices of e_Π are a pair of mates in Π. Since $p(G) = \min_{x\in V(G)} d(x, x') \leq 1$, we have $p(G) = 1$ as desired.

(ii) Necessity. Let $p(G) = k > 1$. Then G is k-pairable. Let Π be a k-pair partition of G. Then G has a pair of vertices x, x' such that x' is the mate of x in Π and $d(x, x') = k$. Assume that P is a shortest path joining x and x' in G. Then P can be written as $xx_1 \cdots x_{k-1}x'$. Assume that x_i' is the mate of x_i in Π for $1 \leq i \leq k-1$. Then $d(x_i, x_i') \geq k$ for $1 \leq i \leq k-1$.

If $k = 2$, then $P = xx_1x'$, and the fact that G has a strongly induced 4-cycle $xx_1x'x_1'$ can be easily seen as follows: x_1' cannot be any vertex of P since $d(x_1, x_1') \geq 2$ but x_1' is adjacent with both x and x' since x_1 is adjacent with both x and x'.

If $k > 2$, then $d(x_1, x_1') \geq k$ implies that x_1' cannot be any vertex of P, and x_1' is not adjacent with any vertex of $P \setminus \{x'\}$; but x_1' adj x' since x_1 adj x. Continuing this way, we can see that for $2 \leq i \leq k-2$, x_i' cannot be any vertex of $P \cup \{x_1', x_2', \ldots, x_{i-1}'\}$, and x_i' is not adjacent with any vertex of $P \cup \{x_1', x_2', \ldots, x_{i-2}'\}$ since $d(x_i, x_i') \geq k$; but x_i' adj x_{i-1}' since x_i adj x_{i-1}. Similarly, x_{k-1}' cannot be any vertex of $P \cup \{x_1', x_2', \ldots, x_{k-2}'\}$, and x_{k-1}' is not adjacent with any vertex of $P \cup \{x_1', x_2', \ldots, x_{k-3}'\}$ since $d(x_{k-1}, x_{k-1}') \geq k$; but x_{k-1}' must be adjacent with both x_{k-2}' and x since x_{k-1} is adjacent with both x_{k-2} and x'. Thus, we can see that $xx_1 \cdots x_{k-2}x_{k-1}x'x_1' \cdots x_{k-2}'x_{k-1}'$ is an induced $2k$-cycle C_Π of G. Since x_i' is the mate of x_i in the k-pair partition Π, $d(x_i, x_i') = k$ for $1 \leq i \leq k-1$. Therefore, C_Π is a strongly induced $2k$-cycle of G.

Sufficiency. Let $k > 1$ and G be a k-pairable graph satisfying the given condition. Then $p(G) \geq k > 1$, and any $p(G)$-pair partition of G is also a k-pair partition of G. Let Π be a $p(G)$-pair partition of G. By the given condition, G has a strongly induced $2k$-cycle C_Π such that the mate of any vertex on C_Π remains on C_Π. Then $p(G) = \min_{x\in V(G)} d(x, x') \leq k$ where x' is the mate of x in Π. Therefore, $p(G) = k$. □

Note: The sufficiency of the above theorem still holds if we replace the strongly induced $2k$-cycle C_Π by just an induced $2k$-cycle.

The following two corollaries can be obtained immediately from Theorem 3.1.

Corollary 3.1. *Let G be a simple graph.*

(i) *If G does not have any strongly induced even cycle, then $p(G) = 0$ or 1.*
(ii) *If G has a strongly induced even cycle, then $p(G) = 0$ or $\frac{l}{2} \leq p(G) \leq \frac{L}{2}$, where l (resp. L) denotes the length of a shortest (resp. longest) strongly induced even cycle of G.*

Example 3.1. The Petersen graph has pair length 0.

Proof. Since the Petersen graph does not have any strongly induced even cycle, its pair length must be 0 or 1 by Corollary 3.1. It is easy to verify that any two adjacent vertices cannot be mates to each other for any 1-pair partition. So its pair length must be 0. □

Corollary 3.2. *Let G be a graph with $p(G) = r(G) > 1$. If u is a central vertex of G, then for any $p(G)$-pair partition of G, its mate u' is also a central vertex of G, and G has a strongly induced cycle C of length $2 \cdot r(G)$ such that both u and u' are on C.*

Proof. By Lemmas 2.1 and 2.2 in [4], if u is a central vertex of a graph G and $p(G) = r(G) > 1$, then its mate u' is also a central vertex of G and $d(u, u') = r(G)$ for any $p(G)$-pair partition of G. Then the corollary follows from the proof of Theorem 3.1. □

In the next main theorem, we determine the minimum order of an r-regular prime graph with pair length k, and in the proof we construct such a graph with the minimum order for any given integers $r, k \geq 2$ (excluding $r = k = 2$). From now on, we will use the notation $N_e[\alpha]$ to denote the minimum even number $\geq \alpha$.

Theorem 3.2. *Let $\mathbb{G}(r, k)$ be the set of r-regular prime graphs with pair length k, where $r, k \geq 2$ are integers.*

(i) *If $k = r = 2$, then $\mathbb{G}(2, 2)$ is an empty set.*
(ii) *If $k = 2$ and $r > 2$, then*

$$\min_{G\in\mathbb{G}(r,2)} \{|V(G)|\} = \begin{cases} 12, & \text{if } r = 3; \\ r+2, & \text{if } r \text{ is even}; \\ r+3, & \text{if } r \equiv 1 \pmod 4; \\ r+5, & \text{if } r \equiv 3 \pmod 4 \text{ and } r > 3. \end{cases}$$

(iii) *If $k > 2$ and $r \geq 2$, then*

$$\min_{G\in\mathbb{G}(r,k)} \{|V(G)|\} = \begin{cases} N_e\left[\dfrac{2k(r+1)}{3}\right] + 2, & \text{if } k = 3m+2 \text{ and } r = 3n \text{ where } n \text{ is odd, or} \\ & \text{if } k = 3m+1 \text{ and } r = 3n+1 \text{ where } n \text{ is even}; \\ N_e\left[\dfrac{2k(r+1)}{3}\right], & \text{otherwise}. \end{cases}$$

The proof for (i) is easy. Note that a connected 2-regular graph with pair length 2 is just a 4-cycle, which is not a prime graph. So, $\mathbb{G}(2, 2)$ is an empty set.

The proof for (ii) and (iii) is quite long. So we prove them in two Sections 3.1 and 3.2, respectively. In the proof, we not only give the minimum order of a graph in $\mathbb{G}(r, k)$, but also provide a construction of such a graph with the minimum order.

From our proof of Theorem 3.2, it is clear that the minimum order of an r-regular prime graph G with pair length k for $r, k \geq 2$ (except $r = k = 2$) is still valid when the restriction "prime" on the considered graphs is removed. So we obtain the following theorem which determines the minimum order of an r-regular graph with pair length k for any integers $r, k \geq 2$.

Theorem 3.3. *Let $\Gamma(r, k)$ be the set of r-regular graphs with pair length k, where $r, k \geq 2$ are integers.*

(i) *If $k = r = 2$, then $\min_{G\in\Gamma(r,k)}\{|V(G)|\} = 4$.*

(ii) *If $k = 2$ and $r > 2$, then*

$$\min_{G\in\Gamma(r,2)}\{|V(G)|\} = \begin{cases} 12, & \text{if } r = 3; \\ r+2, & \text{if } r \text{ is even}; \\ r+3, & \text{if } r \equiv 1 \pmod 4; \\ r+5, & \text{if } r \equiv 3 \pmod 4 \text{ and } r > 3. \end{cases}$$

(iii) *If $k > 2$ and $r \geq 2$, then*

$$\min_{G\in\Gamma(r,k)}\{|V(G)|\} = \begin{cases} N_e\left\lceil \dfrac{2k(r+1)}{3} \right\rceil + 2, & \text{if } k = 3m+2 \text{ and } r = 3n \text{ where } n \text{ is odd, or} \\ & \text{if } k = 3m+1 \text{ and } r = 3n+1 \text{ where } n \text{ is even}; \\ N_e\left\lceil \dfrac{2k(r+1)}{3} \right\rceil, & \text{otherwise}. \end{cases}$$

3.1. $\mathbb{G}(r, 2)$ where $r > 2$

Part (ii) of Theorem 3.2 is on $\mathbb{G}(r, 2)$ where $r > 2$. The result immediately follows from Proposition 3.1 to 3.4 established below.

Lemma 3.1. *For any integers $r > 2$ and $k \geq 2$, $\min_{G\in\mathbb{G}(r,k)}\{|V(G)|\}$ is an even number $\geq r + 2$.*

Proof. Let G be a graph in $\mathbb{G}(r, k)$ with $r > 2$ and $k \geq 2$. Then $|V(G)|$ is even since G is pairable. It is clear that G is not a complete graph since $k \geq 2$. This implies that $r \leq |V(G)| - 2$. Therefore, $|V(G)|$ is an even number $\geq r + 2$. □

Proposition 3.1. $\min_{G\in\mathbb{G}(3,2)}\{|V(G)|\} = 12$.

Proof. We first show that $\min_{G\in\mathbb{G}(3,2)}\{|V(G)|\} \geq 12$ by contradiction. Suppose that this inequality is not true. Then there is a graph G in $\mathbb{G}(3, 2)$ with $|V(G)| \leq 10$. By Lemma 3.1, we have $|V(G)| \geq 6$. Let Π be a 2-pair partition of G. By Theorem 3.1, G has an induced 4-cycle $C = x_0y_0x_0'y_0'$ such that $\{x_0, x_0'\}$ and $\{y_0, y_0'\}$ are pairs of mates in Π. Let $A = V(G - C)$. Then $2 \leq |A| \leq 6$. Let A_i be the subset of A each vertex of which is adjacent with exactly i vertices of C. Then $A = A_0 \cup A_1 \cup A_2 \cup A_3$ since G is 3-regular. Clearly, each vertex of the 4-cycle C is adjacent with exactly one vertex of A, and so the number of edges between C and A is 4. Thus we have

$$\begin{cases} 2 \leq |A_0| + |A_1| + |A_2| + |A_3| \leq 6 & (1) \\ |A_1| + 2|A_2| + 3|A_3| = 4. & (2) \end{cases}$$

Note that the mate of a vertex of A_i in Π must be also in the same A_i. Then each $|A_i|$ $(0 \leq i \leq 3)$ is an even number. Hence, (2) implies that $|A_3| = 0$, and that there are only two possible cases: either $|A_1| = 4$ and $|A_2| = 0$, or $|A_1| = 0$ and $|A_2| = 2$. Below we show that both cases lead to contradictions.

Case 1. $|A_1| = 4$ and $|A_2| = 0$.

By formula (1), $|A_0| \leq 2$. It is easy to see that $|A_0| \neq 0$ (Otherwise, G must be the 3-cube. It contradicts the assumption that G is prime.) Hence, $|A_0| = 2$. Clearly, the two vertices in A_0 are not adjacent since they are mates to each other in Π and $p(G) = 2 > 1$. Then since G is 3-regular, each vertex of A_0 must be adjacent with 3 vertices of A_1, and so the number of edges between A_0 and A_1 is 6. Note that the 4 vertices in A_1 must be 2 pairs of mates in Π. Then, in one pair, say $\{x_1, x_1'\}$, each vertex is adjacent with both vertices of A_0, while in the other pair $\{x_2, x_2'\}$, each vertex is adjacent with just one vertex of A_0. It follows that the only possibility for x_2 and x_2' to have degree 3 is that they are adjacent. This contradicts the condition that $p(G) = 2$.

Case 2. $|A_1| = 0$ and $|A_2| = 2$.

By formula (1), $|A_0| \leq 4$. Since $|A_2| = 2$ and $p(G) = 2$, the two vertices of A_2 are a pair of mates and so they are not adjacent in G. Thus, each vertex of A_2 must be adjacent with exactly one vertex of A_0, since its degree is 3. This implies that $|A_0|$ is a positive even number. Since each vertex of A_0 has degree 3, $|A_0| > 2$. Hence, $|A_0| = 4$. Then we may write

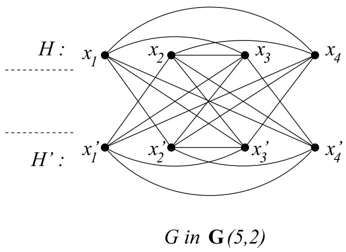


Fig. 2. $\min_{G\in\mathbb{G}(4n+1,2)}\{|V(G)|\} = 4n+4$.

$A_0 = \{y, y', z, z'\}$ where $\{y, y'\}$ and $\{z, z'\}$ are two pairs of mates in Π. Further, we may assume that $A_2 = \{x, x'\}$, a pair of mates in Π, where x is adjacent with $y \in A_0$, and x' is adjacent with $y' \in A_0$. Then, the only possibility for z and z' to attain degree 3 is that they are adjacent. This contradicts the condition that $p(G) = 2$.

Thus we have proved that $\min_{G\in\mathbb{G}(3,2)}\{|V(G)|\} \geq 12$. To show that only the equality holds, we construct a 3-regular graph G with 12 vertices (see Fig. 1(a)) and prove $G \in \mathbb{G}(3, 2)$ as follows. By Lemma 2.2, G is prime since the edge x_2y_2 is not contained in any 4-cycle of G. It is also easy to see that G has a 2-pair partition $\bigcup_{0\leq i\leq 2}(\{x_i, x_i'\} \cup \{y_i, y_i'\})$. We prove that $p(G) = 2$ by contradiction. Suppose that $p(G) > 2$ and Π is a $p(G)$-pair partition of G. Then the mate of y_0 in Π must be one of y_1', x_2', y_2'. But $y_1'x_2'y_2'$ is a triangle in G, and y_0 is not contained in any triangle. This is a contradiction. Therefore, $p(G) = 2$, and so $G \in \mathbb{G}(3, 2)$. □

Proposition 3.2. *For any integer $n \geq 2$, $\min_{G\in\mathbb{G}(2n,2)}\{|V(G)|\} = 2n + 2$.*

Proof. By Lemma 3.1, $\min_{G\in\mathbb{G}(2n,2)}\{|V(G)|\} \geq 2n + 2$. Let G be a graph obtained from K_{2n+2} by deleting a perfect matching M. Then it is not difficult to see that G is $2n$-regular and $p(G) = 2$. We claim that G is strongly triangulated. Consider any two vertices x, y of G. If x and y are adjacent in G, then $xx', yy' \in M$. Note that $|V(G)| = 2n + 2 \geq 6$ since $n \geq 2$. Then there must be a vertex $z \in V(G) \setminus \{x, y, x', y'\}$ such that xyz is a triangle. If x and y are not adjacent in G, then each of them must be adjacent with all vertices of $V(G) \setminus \{x, y\}$. Clearly, there exist vertices $u, v \in V(G) \setminus \{x, y\}$ such that uv is an edge of G. So x and y are joined by two triangles xuv and uvy sharing a common edge uv. Hence, G is strongly triangulated. By Lemma 2.3, G is prime, and so $G \in \mathbb{G}(2n, 2)$. Then the proof is complete since $|V(G)| = 2n + 2$. □

Proposition 3.3. *For any integer $n \geq 1$, $\min_{G\in\mathbb{G}(4n+1,2)}\{|V(G)|\} = 4n + 4$.*

Proof. By Lemma 3.1, $\min_{G\in\mathbb{G}(4n+1,2)}\{|V(G)|\} \geq 4n + 4$. We construct a graph G in $\mathbb{G}(4n + 1, 2)$ with $4n + 4$ vertices as follows. (See Fig. 2 for the case when $n = 1$.) Let H and H' be two isomorphic copies of $K_{2n+2} - M$, where M is a perfect matching of the complete graph K_{2n+2}. For any vertex x of H, let x' denote its isomorphic image in H'. Then the desired G is constructed by adding new edges between H and H' such that each vertex x of H is adjacent with all vertices of $V(H') \setminus \{x'\}$. Clearly, G is $(4n + 1)$-regular. It is also easy to see that G has a 2-pair partition with $\bigcup_{x\in V(H)}\{x, x'\}$. Since any two vertices in G has distance at most 2, we immediately get $p(G) = 2$ by Lemma 2.1. Note that G is the complement of $n + 1$ disjoint 4-cycles. Then it is not difficult to see that G is strongly triangulated, and so G is prime by Lemma 2.3. This completes the proof. □

Proposition 3.4. *For any integer $n \geq 1$, $\min_{G\in\mathbb{G}(4n+3,2)}\{|V(G)|\} = 4n + 8$.*

Proof. We first show that $\min_{G\in\mathbb{G}(4n+3,2)}\{|V(G)|\} \geq 4n + 8$ by contradiction. Suppose that there exists a graph G in $\mathbb{G}(4n + 3, 2)$ with $|V(G)| < 4n + 8$. By Lemma 3.1, $|V(G)| = 4n + 6$. Note that the complement of G is a 2-regular graph containing a perfect matching and having an automorphism which swaps the vertices of each matching edge. Then it must be a union of 4-cycles, which conflicts the fact that the order of G is 2 mod 4. Thus, we have shown that $\min_{G\in\mathbb{G}(4n+3,2)}\{|V(G)|\} \geq 4n + 8$.

Next, we show that only the equality holds, by constructing a graph G in $\mathbb{G}(4n + 3, 2)$ with $4n + 8$ vertices. It is well known (for example, see [3]) that every complete graph with even order is 1-factorable. So K_{2n+4} has $2n+3$ pairwise disjoint perfect matchings. Let H and H' be two isomorphic copies of $K_{2n+4} - \bigcup_{1\leq i\leq 3} M_i$, where M_i for $1 \leq i \leq 3$ are three pairwise disjoint perfect matchings of K_{2n+4}. For any vertex x in H, let x' be its corresponding isomorphic vertex in H'. Let G be a graph constructed from H and H' by joining each x of H to all vertices of $V(H') \setminus \{x'\}$.

Finally, we show that $G \in \mathbb{G}(4n + 3, 2)$. It is easy to see that G is $(4n + 3)$-regular, and that $\bigcup_{x\in V(H)}\{x, x'\}$ gives a 2-pair partition of G with $d(x, x') = 2$. Clearly, any two vertices in G have distance at most 2. So we get $p(G) = 2$ by Lemma 2.1. Below we show that G is prime by contradiction. Otherwise, we may assume that G is the Cartesian product of two non-trivial graphs, say H_1 and H_2, with $1 < |V(H_1)| \leq |V(H_2)|$. Then every vertex of G must have at least $(|V(H_1)| - 1)(|V(H_2)| - 1)$ non-neighbors. But in fact every vertex of G has exactly 4 non-neighbors, so $(|V(H_1)|, |V(H_2)|)$ must be equal to either (2, 4) or (2, 5) or (3, 3). Then $|V(G)|$ must be 8, 10 or 9, contradicting $|V(G)| = 4n + 8 \geq 12$. Thus, G is prime. This completes the proof. □

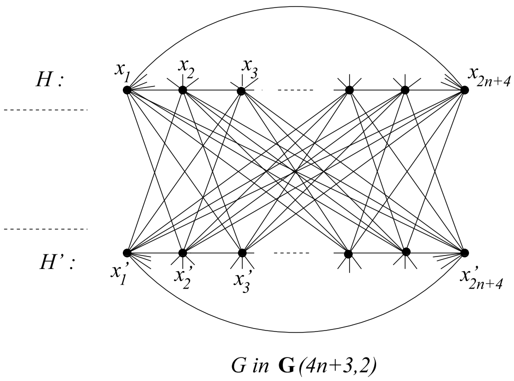


Fig. 3. $\min_{G\in\mathbb{G}(4n+3,2)}\{|V(G)|\} = 4n + 8$.

Remark. The graph G constructed in Proposition 3.4 can be depicted as in Fig. 3, where subgraphs H and H' are Hamiltonian. The reason is given below. When $n = 1$, it is easy to find three pairwise disjoint perfect matchings M_i $(1 \le i \le 3)$ of K_6 such that $H \cong H' \cong K_6 - \bigcup_{1\le i\le 3} M_i$ is an induced 6-cycle. When $n > 1$, each of H and H' is regular of degree $2n$ which is at least half of the vertex number $2n + 4$. So H and H' are Hamiltonian by the well-known Dirac's Theorem (see [3]). From Fig. 3, it is easy to see that the constructed graph G is not only prime but also strongly triangulated.

3.2. $\mathbb{G}(r, k)$ where $r \ge 2$ and $k > 2$

Part (iii) of Theorem 3.2 is on $\mathbb{G}(r, k)$ where $r \ge 2$ and $k > 2$. It follows immediately from Propositions 3.5–3.7 established in this subsection.

Lemma 3.2. *For integers $r \ge 2$ and $k > 2$, $\min_{G\in\mathbb{G}(r,k)}\{|V(G)|\} \ge N_e\left[\frac{2k(r+1)}{3}\right]$.*

Proof. If $r = 2$, then it holds trivially since the only graph in $\mathbb{G}(2, k)$ is the $2k$-cycle. Assume that $r > 2$ and G is a graph in $\mathbb{G}(r, k)$. By Theorem 3.1, G contains a strongly induced $2k$-cycle C. Since G is r-regular, each vertex of C is adjacent with $r - 2$ vertices of $G - C$. Hence, the number of edges between C and $G - C$ is $2k \cdot (r - 2)$. On the other hand, since C is a strongly induced cycle with length $2k \ge 6$, each vertex of $G - C$ is adjacent with at most three vertices of C. Hence, $2k \cdot (r - 2) \le (|V(G)| - 2k) \cdot 3$, and so $|V(G)| \ge \frac{2k(r+1)}{3}$. It follows that $\min_{G\in\mathbb{G}(r,k)}\{|V(G)|\} \ge N_e\left[\frac{2k(r+1)}{3}\right]$, since $|V(G)|$ is even. $\square$

Now, we first consider $\mathbb{G}(r, k)$ under the condition that $r \equiv 2 (\mathrm{mod}\ 3)$ or $k \equiv 0 (\mathrm{mod}\ 3)$ with $r \ge 2$ and $k > 2$.

Proposition 3.5. *Let $r \ge 2$ and $k > 2$ be integers. If $r \equiv 2(\mathrm{mod}\ 3)$ or $k \equiv 0(\mathrm{mod}\ 3)$, then*

$$\min_{G\in\mathbb{G}(r,k)}\{|V(G)|\} = N_e\left[\frac{2k(r+1)}{3}\right] = \frac{2k(r+1)}{3}.$$

Proof. If $r = 2$, then it holds trivially since the only graph in $\mathbb{G}(2, k)$ is the $2k$-cycle. So we may assume that both $r, k > 2$. By Lemma 3.2, we only need to construct a graph in $\mathbb{G}(r, k)$ with $\frac{2k(r+1)}{3}$ vertices (as depicted in Fig. 4).

Case 1. $r \equiv 2(\mathrm{mod}\ 3)$. We may let $r = 3n + 2$ for some integer $n \ge 1$. Then $\frac{2k(r+1)}{3} = 2k(n + 1)$. We construct a graph in $\mathbb{G}(3n+2, k)$ with $2k(n+1)$ vertices as follows. Start with a $2k$-cycle $C = a_1a_2\cdots a_ka'_1a'_2\cdots a'_k$. Replace each vertex a_i (resp. a'_i) of C by a clique K_{n+1} and denote it as A_i (resp. A'_i) for $1 \le i \le k$. Then, we join these cliques in the following way: for any two adjacent vertices of C, the two corresponding cliques are joined by adding all possible edges between their vertex sets. Thus we obtain a $(3n + 2)$-regular graph G with $2k(n + 1)$ vertices. It is easy to see that G is strongly triangulated, and so G is prime by Lemma 2.3. Since the radius of G is $r(G) = k$, we have $p(G) \le k$ by Lemma 2.1. Moreover, G has a k-pair partition such that the mate of each vertex in A_i is a vertex in A'_i for $1 \le i \le k$. Then we get $p(G) = k$. Therefore, $G \in \mathbb{G}(3n + 2, k)$.

Case 2. $k \equiv 0(\mathrm{mod}\ 3)$. We may let $k = 3m$ for some integer $m \ge 1$. Then $\frac{2k(r+1)}{3} = 2m(r + 1)$. We construct a graph in $\mathbb{G}(r, 3m)$ with $2m(r+1)$ vertices as follows. Start with a $2(3m)$-cycle $C = a_1a_2\cdots a_{3m}a'_1a'_2\cdots a'_{3m}$. Add $2m$ cliques K_{r-2} and denote them as X_j and X'_j for $1 \le j \le m$. Then add edges between X_j (resp. X'_j) and C such that each vertex of X_j (resp. X'_j) is adjacent with 3 vertices of C: $a_{3j-2}, a_{3j-1}, a_{3j}$ (resp. $a'_{3j-2}, a'_{3j-1}, a'_{3j}$) for $1 \le j \le m$. Then we obtain an r-regular graph G with $2m(r+1)$ vertices. It is easy to see that G does not have any induced 4-cycle. So G is a prime graph by Lemma 2.2. Since the radius of G is $r(G) = 3m$, we have $p(G) \le 3m$ by Lemma 2.1. Moreover, G has a $3m$-pair partition such that the mate of a_i is a'_i for $1 \le i \le 3m$ and the mate of each vertex in X_j is a vertex in X'_j for $1 \le j \le m$. Then we get $p(G) = 3m$. Therefore, $G \in \mathbb{G}(r, 3m)$. This completes the proof. $\square$

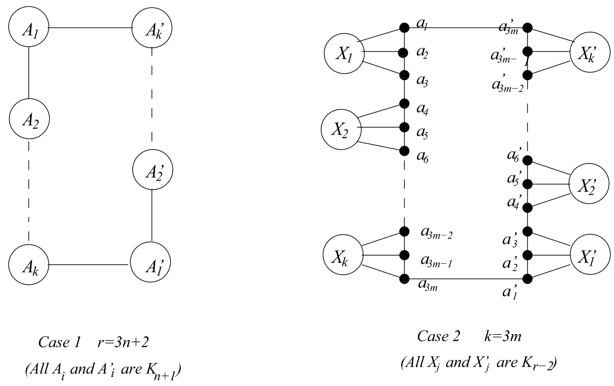


Fig. 4. Graphs constructed for Proposition 3.5.

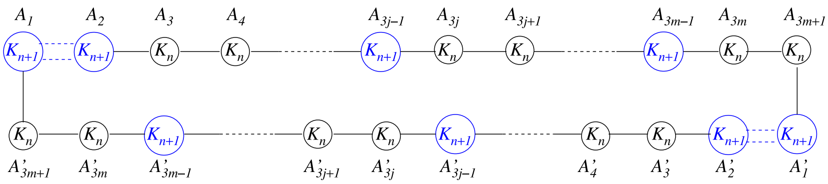


Fig. 5. A graph constructed in $\mathbb{G}(3n, 3m + 1)$.

In the rest of the paper, we will distinguish the remaining cases of $\mathbb{G}(r, k)$ $(r, k > 2)$ according to $r \equiv 0 (\text{mod } 3)$ or $r \equiv 1(\text{mod } 3)$. The results are given in the two propositions below.

Proposition 3.6. *For integers $n \geq 1$ and $k > 2$,*

$$\min_{G\in\mathbb{G}(3n,k)}\{|V(G)|\} = \begin{cases} N_e\left[\dfrac{2k(3n+1)}{3}\right]+2, & \text{if } n \text{ is odd and } k \equiv 2(\text{mod } 3);\\ N_e\left[\dfrac{2k(3n+1)}{3}\right], & \text{otherwise}. \end{cases}$$

Proof. By Lemma 3.2, $\min_{G\in\mathbb{G}(3n,k)}\{|V(G)|\} \geq N_e\left[\frac{2k(3n+1)}{3}\right]$. We distinguish three cases.

Case 1. $k \equiv 0$ (mod 3). Then $\min_{G\in\mathbb{G}(3n,k)}\{|V(G)|\} = N_e\left[\frac{2k(3n+1)}{3}\right]$ by Proposition 3.5.

Case 2. $k \equiv 1$ (mod 3). Then $k = 3m + 1$ and $N_e\left[\frac{2k(3n+1)}{3}\right] = 2(3m + 1)n + 2m + 2$ for some integer $m \geq 1$. To show that $\min_{G\in\mathbb{G}(3n,k)}\{|V(G)|\} = N_e\left[\frac{2k(3n+1)}{3}\right]$, we only need to construct a graph in $\mathbb{G}(3n, 3m + 1)$ with $2(3m + 1)n + 2m + 2$ vertices.

Start with a $2(3m + 1)$-cycle $C = a_1a_2 \cdots a_{3m+1}a'_1a'_2 \cdots a'_{3m+1}$. Replace each vertex a_i (resp. a'_i) $(1 \leq i \leq 3m + 1)$ of C by a clique A_i (resp. A'_i) such that

$$A_i = A'_i = \begin{cases} K_n, & \text{if } i = 3j \text{ or } i = 3j + 1 \text{ for } 1 \leq j \leq m,\\ K_{n+1}, & \text{otherwise.} \end{cases}$$

Then, for any two adjacent vertices of C, we join the two corresponding cliques according to the following rules: If the two corresponding cliques are both K_{n+1}, then we add all possible edges but an $(n+1)$-matching between their vertex sets; (Note that the two corresponding K_{n+1} cliques are pairs $\{A_1, A_2\}$ or $\{A'_1, A'_2\}$.) Otherwise, we add all possible edges between them. See Fig. 5.

Thus, we obtain a $3n$-regular graph G with $2(3m + 1)n + 2m + 2$ vertices. It is clear that any edge between A_3 and A_4 is not contained in any induced 4-cycle of G. So G is prime by Lemma 2.2. Since the radius of G is $r(G) = 3m + 1$, we have $p(G) \leq 3m + 1$ by Lemma 2.1. Moreover, G has a $(3m + 1)$-pair partition such that the mate of a vertex in A_i is a vertex in A'_i for $1 \leq i \leq 3m + 1$. Then $p(G) = 3m + 1$ and so $G \in \mathbb{G}(3n, 3m + 1)$.

Case 3. $k \equiv 2$ (mod 3). Then $k = 3m + 2$ and $N_e\left[\frac{2k(3n+1)}{3}\right] = 2(3m + 2)n + 2m + 2$ for some integer $m \geq 1$. We distinguish two subcases according to the parity of n.

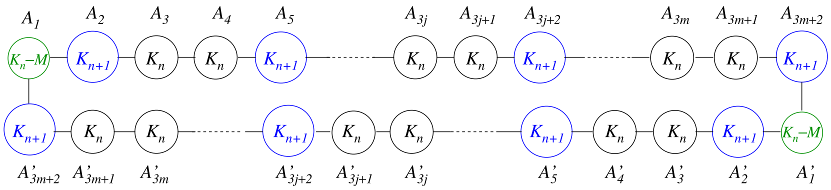


Fig. 6. A graph constructed in $\mathbb{G}(3n, 3m+2)$ when n is even.

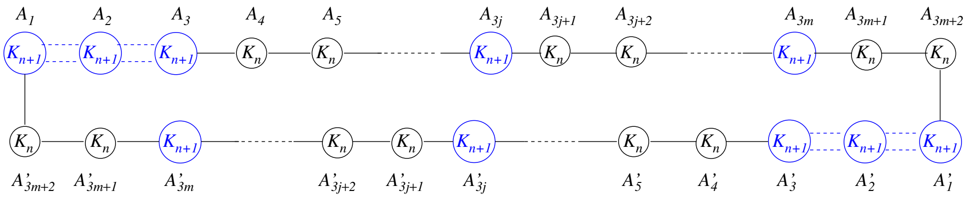


Fig. 7. A graph constructed in $\mathbb{G}(3n, 3m+2)$ when n is odd.

Subcase 3.1. n is even. To show that $\min_{G\in\mathbb{G}(3n,k)}\{|V(G)|\} = N_e\left[\frac{2k(3n+1)}{3}\right]$, we only need to construct a graph in $\mathbb{G}(3n, 3m+2)$ with $2(3m+2)n + 2m + 2$ vertices.

Start with a $2(3m+2)$-cycle $C = a_1a_2\cdots a_{3m+2}a'_1a'_2\cdots a'_{3m+2}$. Replace each vertex a_i (resp. a'_i) $(1 \le i \le 3m+2)$ of C by a graph A_i (resp. A'_i) as follows.

$$A_i = A'_i = \begin{cases} K_n - M, & \text{if } i = 1,\\ K_n, & \text{if } i = 3j \text{ or } i = 3j+1 \text{ for } 1 \le j \le m,\\ K_{n+1}, & \text{otherwise,}\end{cases}$$

where M is a perfect matching of K_n.

Then, for any two adjacent vertices of C, add all possible edges between the two vertex sets of the corresponding graphs. See Fig. 6.

Thus, we obtain a $3n$-regular graph G with $2(3m+2)n + 2m + 2$ vertices. It is easy to see that any edge between A_2 and A_3 is not contained in any induced 4-cycle of G. So G is prime by Lemma 2.2. Since the radius of G is $r(G) = 3m+2$, we have $p(G) \le 3m+2$ by Lemma 2.1. Moreover, G has a $(3m+2)$-pair partition such that the mate of a vertex in A_i is a vertex in A'_i for $1 \le i \le 3m+2$. We get $p(G) = 3m+2$ and so $G \in \mathbb{G}(3n, 3m+2)$.

Subcase 3.2. n is odd. To show that $\min_{G\in\mathbb{G}(3n,k)}\{|V(G)|\} = N_e\left[\frac{2k(3n+1)}{3}\right] + 2$, we only need to construct a graph in $\mathbb{G}(3n, 3m+2)$ with $2(3m+2)n + 2m + 4$ vertices, since we have the following claim.

Claim A. *Let $m, n \ge 1$ be integers. If n is odd, then*

$$\min_{G\in\mathbb{G}(3n,3m+2)}\{|V(G)|\} \ne 2(3m+2)n + 2m + 2.$$

(Proof of Claim A will be given at the end.)

A desired graph can be constructed as follows. Start with a $2(3m+2)$-cycle $C = a_1a_2\cdots a_{3m+2}a'_1a'_2\cdots a'_{3m+2}$. Replace each vertex a_i (resp. a'_i) $(1 \le i \le 3m+2)$ of C by a clique A_i (resp. A'_i) such that

$$A_i = A'_i = \begin{cases} K_n, & \text{if } i = 3j+1 \text{ or } i = 3j+2 \text{ for } 1 \le j \le m,\\ K_{n+1}, & \text{otherwise.}\end{cases}$$

Then, for any two adjacent vertices of C, we join the two corresponding cliques according to the following rules: If the two corresponding cliques are both K_{n+1}, then we add all possible edges but an $(n+1)$-matching between the two vertex sets of the cliques; (Note that the two corresponding K_{n+1} cliques are the pairs $\{A_1, A_2\}$, $\{A_2, A_3\}$, $\{A'_1, A'_2\}$, and $\{A'_2, A'_3\}$.) Otherwise, we add all possible edges between them. See Fig. 7.

Thus we obtain a $3n$-regular graph G with $2(3m+2)n + 2m + 4$ vertices. It is clear that any edge between A_4 and A_5 is not contained in any induced 4-cycle of G. So G is prime by Lemma 2.2. Since the radius of G is $r(G) = 3m+2$, we have $p(G) \le 3m+2$ by Lemma 2.1. Moreover, G has a $(3m+2)$-pair partition such that the mate of a vertex in A_i is a vertex in A'_i for $1 \le i \le 3m+2$. Hence, $p(G) = 3m+2$ and so $G \in \mathbb{G}(3n, 3m+2)$. $\square$

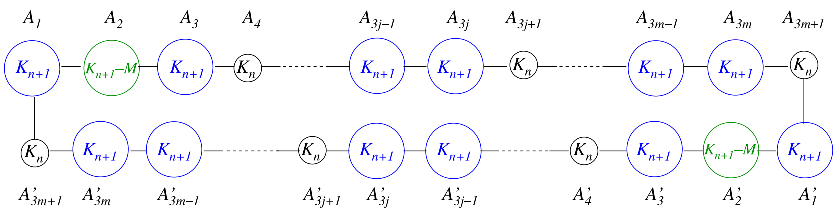


Fig. 8. A graph constructed in $\mathbb{G}(3n+1, 3m+1)$ when n is odd.

Proposition 3.7. *Let $n \geq 1$ and $k > 2$ be integers. Then*

$$\min_{G\in\mathbb{G}(3n+1,k)}\{|V(G)|\} = \begin{cases} N_e\left[\dfrac{2k(3n+2)}{3}\right]+2, & \text{if } n \text{ is even and } k\equiv 1 (\text{mod } 3);\\ N_e\left[\dfrac{2k(3n+2)}{3}\right], & \text{otherwise}. \end{cases}$$

Proof. By Lemma 3.2, $\min_{G\in\mathbb{G}(3n+1,k)}\{|V(G)|\} \geq N_e\left[\frac{2k(3n+2)}{3}\right]$. We distinguish three cases.

Case 1. $k \equiv 0 \pmod 3$. Then $\min_{G\in\mathbb{G}(3n+1,k)}\{|V(G)|\} = N_e\left[\frac{2k(3n+2)}{3}\right]$ by Proposition 3.5.

Case 2. $k \equiv 1 \pmod 3$. Then $k = 3m+1$ and $N_e\left[\frac{2k(3n+2)}{3}\right] = 2(3m+1)n+4m+2$ for some integer $m \geq 1$. We distinguish two subcases according to the parity of n.

Subcase 2.1. n is odd. To show that $\min_{G\in\mathbb{G}(3n+1,k)}\{|V(G)|\} = N_e\left[\frac{2k(3n+2)}{3}\right]$, we only need to construct a graph in $\mathbb{G}(3n+1, 3m+1)$ with $2(3m+1)n+4m+2$ vertices.

A desired graph can be constructed as follows. Start with a $2(3m+1)$-cycle $C = a_1a_2\cdots a_{3m+1}a'_1a'_2\cdots a'_{3m+1}$. Replace each vertex a_i (resp. a'_i) $(1 \leq i \leq 3m+1)$ of C by a graph A_i (resp. A'_i) in the following way.

$$A_i = A'_i = \begin{cases} K_n, & \text{if } i = 3j+1 \text{ for } 1 \leq j \leq m,\\ K_{n+1}-M, & \text{if } i = 2,\\ K_{n+1}, & \text{otherwise}, \end{cases}$$

where M is a perfect matching of K_{n+1}.

Then, for any two adjacent vertices of C, add all possible edges between the two vertex sets of the corresponding graphs. See Fig. 8.

Thus we obtain a $(3n+1)$-regular graph G with $2(3m+1)n+4m+2$ vertices. It is easy to see that any edge between A_3 and A_4 is not contained in any induced 4-cycle of G. So G is prime by Lemma 2.2. Since the radius of G is $r(G) = 3m+1$, we have $p(G) \leq 3m+1$ by Lemma 2.1. Moreover, G has a $(3m+1)$-pair partition such that the mate of a vertex in A_i is a vertex in A'_i for $1 \leq i \leq 3m+1$. Then we get $p(G) = 3m+1$ and so $G \in \mathbb{G}(3n+1, 3m+1)$.

Subcase 2.2. n is even. To show that $\min_{G\in\mathbb{G}(3n+1,k)}\{|V(G)|\} = N_e\left[\frac{2k(3n+2)}{3}\right]+2$, we only need to construct a graph in $\mathbb{G}(3n+1, 3m+1)$ with $2(3m+1)n+4m+4$ vertices, since we have the following claim.

Claim B. *Let $m, n \geq 1$ be integers. If n is even, then*

$$\min_{G\in\mathbb{G}(3n+1,3m+1)}\{|V(G)|\} \neq 2(3m+1)n+4m+2.$$

(Proof of Claim B will be given at the end.)

A desired graph can be constructed as follows. Start with a $2(3m+1)$-cycle $C = a_1a_2\cdots a_{3m+1}a'_1a'_2\cdots a'_{3m+1}$. Replace each vertex a_i (resp. a'_i) $(1 \leq i \leq 3m+1)$ of C by a clique A_i (resp. A'_i) for $1 \leq i \leq 3m+1$ as the follows.

If $m = 1$, then $A_i = A'_i = K_{n+1}$ for all $1 \leq i \leq 4$.

If $m > 1$, then

$$A_i = A'_i = \begin{cases} K_n, & \text{if } i = 3j+1 \text{ for } 1 \leq j \leq m-1,\\ K_{n+1}, & \text{otherwise}. \end{cases}$$

For any two adjacent vertices on C, we join the two corresponding cliques according to the following rules: If the two corresponding cliques are the K_{n+1} pairs $\{A_1, A_2\}$, $\{A_{3m}, A_{3m+1}\}$, $\{A'_1, A'_2\}$, and $\{A'_{3m}, A'_{3m+1}\}$, then add all possible edges but an $(n+1)$-matching between the two vertex sets of the cliques; Otherwise, add all possible edges between them. See Fig. 9.

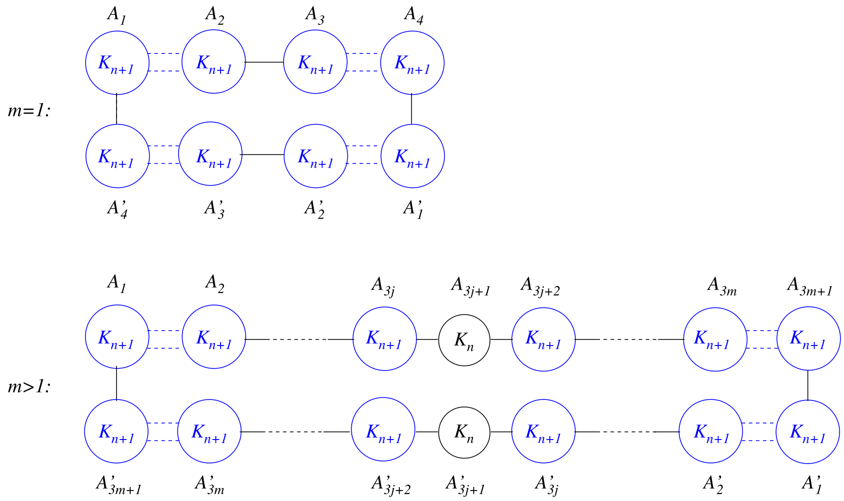


Fig. 9. Graphs constructed in $\mathbb{G}(3n+1, 3m+1)$ when n is even.

Thus we have a $(3n+1)$-regular graph G with $2(3m+1)n+4m+4$ vertices. It is easy to see that any edge between A_2 and A_3 is not contained in any induced 4-cycle of G. So G is prime by Lemma 2.2. Since the radius of G is $r(G) = 3m+1$, we have $p(G) \le 3m+1$ by Lemma 2.1. Moreover, G has a $(3m+1)$-pair partition such that the mate of a vertex in A_i is a vertex in A'_i for $1 \le i \le 3m+1$. Then we get $p(G) = 3m+1$ and so $G \in \mathbb{G}(3n+1, 3m+1)$.

Case 3. $k \equiv 2 (\bmod\ 3)$. Then $k = 3m+2$ and $N_e[\frac{2k(3n+2)}{3}] = 2(3m+2)n+4m+4$ for some integer $m \ge 1$. To show that $\min_{G\in\mathbb{G}(3n+1,k)}\{|V(G)|\} = N_e\left[\frac{2k(3n+2)}{3}\right]$, we only need to construct a graph in $\mathbb{G}(3n+1, 3m+2)$ with $2(3m+2)n+4m+4$ vertices as follows.

Start with a $2(3m+2)$-cycle $C = a_1a_2\cdots a_{3m+2}a'_1a'_2\cdots a'_{3m+2}$. Replace each vertex a_i (resp. a'_i) $(1 \le i \le 3m+2)$ of C by a graph A_i (resp. A'_i) for $1 \le i \le 3m+2$ in the following two different ways according to the parity of n.

Subcase 3.1. If n is even, then

$$A_i = A'_i = \begin{cases} K_{n+2}, & \text{if } i = 2, \\ K_n - M, & \text{if } i = 1, 3, \\ K_n, & \text{if } i = 3j \text{ for } 2 \le j \le m, \\ K_{n+1}, & \text{otherwise,} \end{cases}$$

where M is a perfect matching of K_n.

Subcase 3.2. If n is odd, then

$$A_i = A'_i = \begin{cases} K_{n+1} - M, & \text{if } i = 2, 3, \\ K_n, & \text{if } i = 3j+2 \text{ for } 1 \le j \le m, \\ K_{n+1}, & \text{otherwise,} \end{cases}$$

where M is a perfect matching of K_{n+1}.

Then in both subcases, for any two adjacent vertices on C, add all possible edges between the two vertex sets of the corresponding graphs. See Fig. 10.

Thus we obtain a $(3n+1)$-regular graph G with $2(3m+2)n+4m+4$ vertices. It is clear that G has an edge which is not contained in any induced 4-cycle of G, and so G is prime by Lemma 2.2. Since the radius of G is $r(G) = 3m+2$, we have $p(G) \le 3m+2$ by Lemma 2.1. Moreover, G has a $(3m+2)$-pair partition such that the mate of a vertex in A_i is a vertex in A'_i for $1 \le i \le 3m+2$. Then we get $p(G) = 3m+2$ and so $G \in \mathbb{G}(3n+1, 3m+2)$. This completes the proof. $\square$

Now we give proofs for Claims A and B below.

Claim A. *Let $m, n \ge 1$ be integers. If n is odd, then*

$$\min_{\mathbb{G}(3n,3m+2)}\{|V(G)|\} \ne 2(3m+2)n+2m+2.$$

Proof of Claim A. By contradiction. Suppose that there exists a graph G in $\mathbb{G}(3n, 3m+2)$ with $2(3m+2)n+2m+2$ vertices. We will derive a contradiction.

Let Π be a $(3m+2)$-pair partition of G. By Theorem 3.1, G has a strongly induced $2(3m+2)$-cycle $C = a_1\cdots a_{3m+2}a_{3m+3}\cdots a_{6m+4}$ such that for each i $(1 \le i \le 3m+2)$ the mate of a_i is a_{3m+2+i} in Π. Let $B = V(G-C)$. It is easy

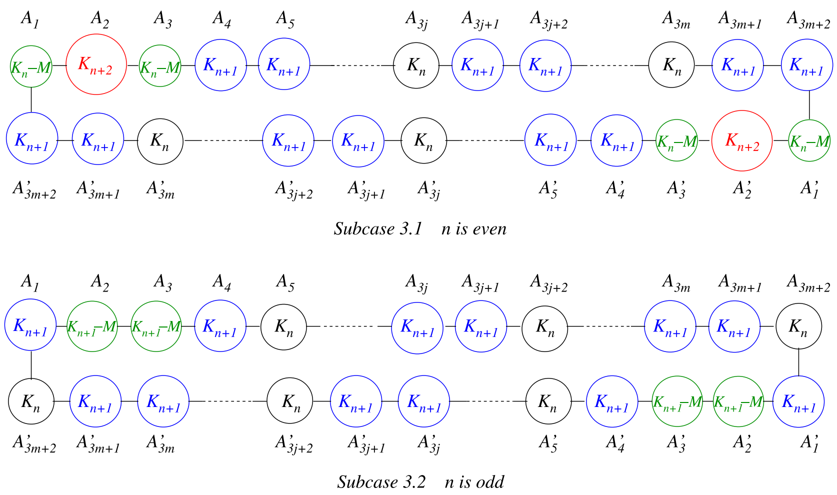


Fig. 10. Graphs constructed in $\mathbb{G}(3n+1, 3m+2)$.

to see that $|B| = 2(3m+2)(n-1)+2m+2$, and that the number of edges between B and C is $2(3m+2)(3n-2)$. Since C is a strongly induced $2(3m+2)$-cycle of G, each vertex of B can be adjacent with at most three vertices of C. Decompose B as $B = B_0 \cup B_1 \cup B_2 \cup B_3$, where B_j $(0 \le j \le 3)$ denotes the subset of B each vertex of which is adjacent with exactly j vertices of C. Then we have the following two equations.

$$\begin{cases} |B_0| + |B_1| + |B_2| + |B_3| = 2(3m+2)(n-1) + 2m + 2 & (1) \\ |B_1| + 2|B_2| + 3|B_3| = 2(3m+2)(3n-2). & (2) \end{cases} \tag{A}$$

Subtracting Eq. (A2) from 3 times Eq. (A1), we get $3|B_0| + 2|B_1| + |B_2| = 2$. Note that each $|B_j|$ is even for $0 \le j \le 3$, since the mate of each vertex of B_j in Π must also be a vertex of B_j. It follows that $|B_0| = |B_1| = 0$ and $|B_2| = 2$. Hence, we have

$$|B_3| = 2(3m+2)(n-1) + 2m. \tag{A3}$$

Assume that $B_2 = \{z, z'\}$. Then z and z' must be mates to each other in Π, and so $d(z, z') \ge 3m+2$. It implies that $n > 1$. Otherwise, $n = 1$, then G is of degree $r = 3n = 3$ and so z adj z', which contradicts the fact that $d(z, z') \ge 3m + 2$.

Note that for each vertex of B_3, its three neighbors on C must be consecutive, since C is a strongly induced cycle. Let $B_{3,i}$ denote the subset of B_3 each vertex of which is adjacent with three consecutive vertices a_{i-1}, a_i, a_{i+1} of C for $1 \le i \le 6m+4$, where the index addition here and later is modulo $6m+4$. Clearly, we have $B_3 = \bigcup_{1 \le i \le 6m+4} B_{3,i}$.

Let $A_i = B_{3,i} \cup \{a_i\}$ for $1 \le i \le 6m+4$. Then $\bigcup_{1 \le i \le 6m+4} A_i = B_3 \cup \{a_i | 1 \le i \le 6m+4\}$. By Eq. (A3) we have

$$\left| \bigcup_{1 \le i \le 6m+4} A_i \right| = 2(3m+2)n + 2m. \tag{A4}$$

It is clear that the mate of each vertex of A_i in Π must be a vertex of A_{3m+2+i}, and vice versa. Therefore, $|A_i| = |A_{3m+2+i}|$ for $1 \le i \le 3m+2$.

In the following we will first find all $|A_i|$, then study the neighbors of z to obtain a contradiction to the assumption that n is odd. Recall that z has exactly two neighbors on C. We distinguish two cases based on whether the two neighbors of z on C are adjacent or not.

Case A1. The two neighbors of z on C are adjacent. Then we may assume that z is adjacent with a_1, a_2 of C, and so z' is adjacent with a_{3m+3}, a_{3m+4} of C. See Fig. 11.

Now we will show that for $1 \le i \le 3m+2$,

$$|A_i| = |A_{3m+2+i}| = \begin{cases} n+1, & \text{if } i = 3j+1 \text{ for } 1 \le j \le m, \\ n, & \text{otherwise.} \end{cases} \tag{A5}$$

Since each vertex of C has degree $3n$ in G, we have the following observations on the size of the union of three consecutive A_i's.

(i) For $i = 1, 2, 3m+3, 3m+4$, $|A_{i-1} \cup A_i \cup A_{i+1}| \ge 3n$, since the vertex a_i can only be adjacent with the vertices from $\{z\} \cup A_{i-1} \cup (A_i \setminus \{a_i\}) \cup A_{i+1}$ or $\{z'\} \cup A_{i-1} \cup (A_i \setminus \{a_i\}) \cup A_{i+1}$;

(ii) For other i, $|A_{i-1} \cup A_i \cup A_{i+1}| \ge 3n+1$, since the vertex a_i can only be adjacent with the vertices from $A_{i-1} \cup (A_i \setminus \{a_i\}) \cup A_{i+1}$.

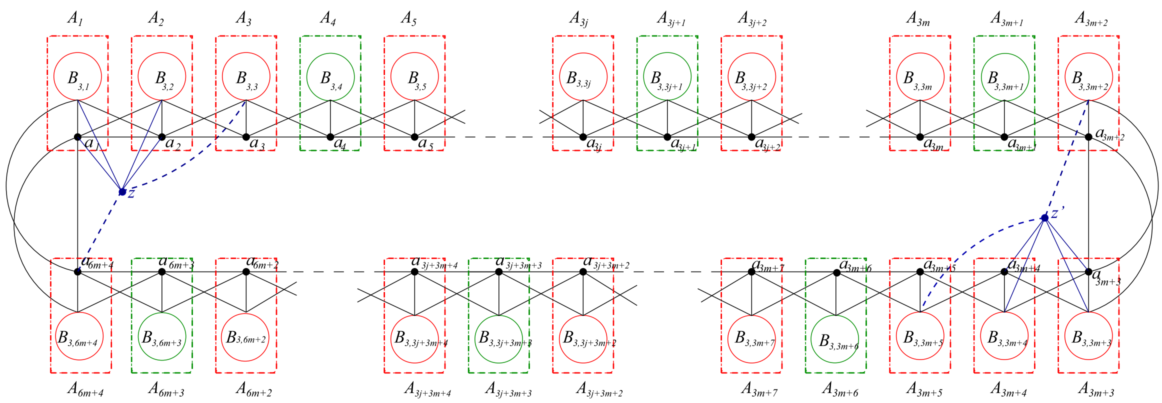


Fig. 11. Illustration for G considered in the proof of Case A1.

By the two observations and Eq. (A4), we get the following inequalities:

$$\begin{aligned}
|A_1| &= \left|\left(\bigcup_{1\le i\le 6m+4} A_i\right) \setminus \left(\bigcup_{1\le k\le 2m+1} (A_{3k-1}\cup A_{3k}\cup A_{3k+1})\right)\right| \\
&\le [2(3m+2)n+2m]-[2m(3n+1)+3n]=n,\\
|A_2| &= \left|\left(\bigcup_{1\le i\le 6m+4} A_i\right) \setminus \left(\bigcup_{1\le k\le 2m} (A_{3k}\cup A_{3k+1}\cup A_{3k+2})\right)\setminus (A_{6m+3}\cup A_{6m+4}\cup A_1)\right| \\
&\le [2(3m+2)n+2m]-[2m(3n+1)+3n]=n,
\end{aligned}$$

and for $1\le j\le m$,

$$\begin{aligned}
|A_{3j}| &= \left|\left(\bigcup_{1\le i\le 6m+4} A_i\right) \setminus \left(\bigcup_{j\le k\le 2m} (A_{3k+1}\cup A_{3k+2}\cup A_{3k+3})\right)\setminus (A_{6m+4}\cup A_1\cup A_2)\setminus\left(\bigcup_{1\le k\le j-1}(A_{3k}\cup A_{3k+1}\cup A_{3k+2})\right)\right| \\
&\le [2(3m+2)n+2m]-[2m(3n+1)+3n]=n.\\
|A_{3j+1}| &= \left|\left(\bigcup_{1\le i\le 6m+4} A_i\right) \setminus \left(\bigcup_{j\le k\le 2m} (A_{3k+2}\cup A_{3k+3}\cup A_{3k+4})\right)\setminus\left(\bigcup_{1\le k\le j}(A_{3k-2}\cup A_{3k-1}\cup A_{3k})\right)\right| \\
&\le [2(3m+2)n+2m]-[(2m-1)(3n+1)+2(3n)]=n+1.\\
|A_{3j+2}| &= \left|\left(\bigcup_{1\le i\le 6m+4} A_i\right) \setminus \left(\bigcup_{j\le k\le 2m-1} (A_{3k+3}\cup A_{3k+4}\cup A_{3k+5})\right)\right. \\
&\qquad \left.\setminus (A_{6m+3}\cup A_{6m+4}\cup A_1)\setminus\left(\bigcup_{1\le k\le j}(A_{3k-1}\cup A_{3k}\cup A_{3k+1})\right)\right| \\
&\le [2(3m+2)n+2m]-[2m(3n+1)+3n]=n.
\end{aligned}$$

Adding all the above inequalities, we get $\sum_{i=1}^{3m+2}|A_i|\le (3m+2)n+m$.

Since $|A_i|=|A_{3m+2+i}|$ for all $1\le i\le 3m+2$, we have $\sum_{i=1}^{6m+4}|A_i|\le 2[(3m+2)n+m]$. Then by Eq. (A4), we see that only the equality holds. Thus, for all the above inequalities on $|A_i|$, only the equalities hold. This proves Eq. (A5).

Since C is a strongly induced $2(3m+2)$-cycle of G, it is easy to see that the neighbors of z are contained in $B_{3,6m+4}\cup A_1\cup A_2\cup B_{3,3}$, and each vertex u of A_1 (resp. A_2) can only be adjacent with vertices from $\{z\}\cup A_{6m+4}\cup(A_1\setminus\{u\})\cup A_2$. (resp. $\{z\}\cup A_1\cup(A_2\setminus\{u\})\cup A_3$).

Note that G is $3n$-regular and $|A_{6m+4}|=|A_1|=|A_2|=n$. Then A_1 must be a clique K_n and each vertex u of A_1 must be adjacent with all $2n+1$ vertices of $\{z\}\cup A_{6m+4}\cup A_2$. Similarly, A_2 must also be a clique K_n and each vertex u of A_2 must be adjacent with all $2n+1$ vertices of $\{z\}\cup A_1\cup A_3$. Thus, z is adjacent with all $2n$ vertices of $A_1\cup A_2$. It follows that z must be adjacent with n vertices from $B_{3,6m+4}\cup B_{3,3}$ since $\deg(z)=3n$.

Below we will show that n is even to obtain a contradiction. Assume that $x\in B_{3,6m+4}$ is adjacent with z. Then x can only be adjacent with vertices from $\{z\}\cup A_{6m+3}\cup(A_{6m+4}\setminus\{x\})\cup A_1$ because C is a strongly induced cycle of G. Since $x\in A_{6m+4}$, we already see in last paragraph that x is adjacent with all n vertices of A_1. Moreover, we can show that x is adjacent with all $n+1$ vertices of A_{6m+3} as follows. Note that each vertex of A_{6m+3} can only be adjacent with vertices from $A_{6m+2}\cup A_{6m+3}\cup A_{6m+4}$, and we have shown that $|A_{6m+2}|=|A_{6m+4}|=n$ and $|A_{6m+3}|=n+1$. Then A_{6m+3} must be a clique K_{n+1} and each vertex of A_{6m+3} is adjacent with all $2n$ vertices of $A_{6m+2}\cup A_{6m+4}$. It implies that the vertex $x\in B_{3,6m+4}$ is adjacent with all $n+1$

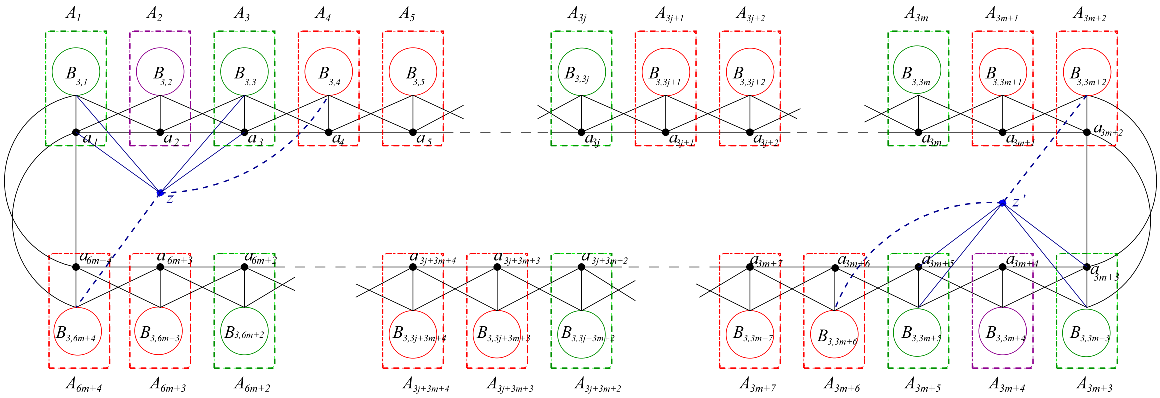


Fig. 12. Illustration for G considered in the proof of Case A2.

vertices of A_{6m+3}. So, x is adjacent with all $2n+2$ vertices of $\{z\}\cup A_{6m+3}\cup A_1$. Then since $\deg(x) = 3n$, the vertex x is adjacent with exactly $n-2$ vertices in A_{6m+4}. Therefore, there is a unique vertex $y \in B_{3,6m+4}\setminus\{x\}$ that is not adjacent with x. It is clear that y must be adjacent with z and all vertices of $A_{6m+4}\setminus\{x, y\}$ since $\deg(y) = 3n$. Similarly, we can show that if $u \in B_{3,3}$ is adjacent with z, then there exists a unique vertex $v \in B_{3,3}\setminus\{u\}$ such that u and v are not adjacent and both of them are adjacent with z and all vertices of $A_3 \setminus \{u, v\}$. It follows that n, the number of neighbors of z in $B_{3,6m+4} \cup B_{3,3}$, is even. This is a contradiction to the assumption that n is odd. Thus we have proved Case A1 of Claim A.

Case A2. The two neighbors of z on C are not adjacent. Then the two must be of distance 2, since C is a strongly induced $2(3m + 2)$-cycle of G. We may assume that z is adjacent with a_1, a_3 of C, and so z' is adjacent with a_{3m+3}, a_{3m+5} of C. See Fig. 12.

Since each vertex of C has degree $3n$ in G, we have the following observations on the size of the union of three consecutive A_i's.

(i) For $i = 1, 3, 3m + 3, 3m + 5$, $|A_{i-1} \cup A_i \cup A_{i+1}| \geq 3n$, since vertex a_i can only be adjacent with vertices from $\{z\} \cup A_{i-1} \cup (A_i \setminus \{a_i\}) \cup A_{i+1}$ or $\{z'\} \cup A_{i-1} \cup (A_i \setminus \{a_i\}) \cup A_{i+1}$;

(ii) For other i, $|A_{i-1} \cup A_i \cup A_{i+1}| \geq 3n + 1$, since vertex a_i can only be adjacent with vertices from $A_{i-1} \cup (A_i \setminus \{a_i\}) \cup A_{i+1}$.

In a way similar to the proof for Case A1, we can get a contradiction in the following steps.

First, we can show that for $1 \leq i \leq 3m + 2$,

$$|A_i| = |A_{3m+2+i}| = \begin{cases} n-1, & \text{if } i = 2,\\ n+1, & \text{if } i = 1 \text{ or } i = 3j \text{ for } 1 \leq j \leq m,\\ n, & \text{otherwise.}\end{cases}$$

Second, the neighbors of z are contained in $B_{3,6m+4} \cup A_1 \cup A_3 \cup B_{3,4}$. We can show that z is adjacent with all $2n+2$ vertices of $A_1 \cup A_3$. Hence, z must be adjacent with $n-2$ vertices in $B_{3,6m+4} \cup B_{3,4}$ since $\deg(z) = 3n$.

Finally, we can show that $n-2$, the number of neighbors of z from $B_{3,6m+4} \cup B_{3,4}$, is even to obtain a contradiction to the assumption that n is odd.

This proves for Case A2, and so the proof of Claim A is complete. □

Claim B. *Let $m, n \geq 1$ be integers. If n is even, then*

$$\min_{G\in\mathbb{G}(3n+1,3m+1)} \{|V(G)|\} \neq 2(3m+1)n + 4m + 2.$$

Proof of Claim B. By contradiction. Suppose that there is a graph G in $\mathbb{G}(3n+1, 3m+1)$ with $2(3m+1)n+4m+2$ vertices. We will derive a contradiction.

Let Π be a $(3m + 1)$-pair partition of G. By Theorem 3.1, G has a strongly induced $2(3m + 1)$-cycle $C = a_1 \cdots a_{3m+1}a_{3m+2} \cdots a_{6m+2}$ such that the mate of a_i is a_{3m+1+i} $(1 \leq i \leq 3m + 1)$ in Π. Let $B = V(G - C)$. Then $|B| = 2(3m + 1)(n - 1) + 4m + 2$. The number of edges between B and C is $2(3m + 1)(3n - 1)$. Since C is a strongly induced $2(3m + 1)$-cycle of G, each vertex of B can be adjacent with at most three vertices of C. Let $B = B_0 \cup B_1 \cup B_2 \cup B_3$, where B_j $(0 \leq j \leq 3)$ is the subset of B each vertex of which is adjacent with exactly j vertices of C. Then we have the following two equations.

$$\begin{cases} |B_0| + |B_1| + |B_2| + |B_3| = 2(3m + 1)(n - 1) + 4m + 2, & (1)\\ |B_1| + 2|B_2| + 3|B_3| = 2(3m + 1)(3n - 1). & (2)\end{cases} \tag{B}$$

Subtracting Eq. (B2) from 3 times Eq. (B1), we get $3|B_0| + 2|B_1| + |B_2| = 2$. Note that each $|B_j|$ is even for $0 \leq j \leq 3$, since the mate of each vertex of B_j in Π is still a vertex of B_j. It follows that $|B_0| = |B_1| = 0$ and $|B_2| = 2$. Hence, we have

$$|B_3| = 2(3m + 1)(n - 1) + 4m. \tag{B3}$$

Assume that $B_2 = \{z, z'\}$ and $B_3 = \bigcup_{1\le i\le 6m+2} B_{3,i}$ where $B_{3,i}$ denotes the subset of B_3 each vertex of which is adjacent with three consecutive vertices a_{i-1}, a_i, a_{i+1} of C.

Let $A_i = B_{3,i} \cup \{a_i\}$ for $1 \le i \le 6m+2$. Then $\bigcup_{1\le i\le 6m+2} A_i = B_3 \cup \{a_i | 1 \le i \le 6m+2\}$ By Eq. (B3), we have

$$\left|\bigcup_{1\le i\le 6m+2} A_i\right| = 2(3m+1)n + 4m. \tag{B4}$$

It is clear that the mate of each vertex of A_i in Π is a vertex of A_{3m+1+i}, and vice versa. Therefore, $|A_i| = |A_{3m+1+i}|$ for $1 \le i \le 3m+1$.

In the following, we will first find $|A_i|$ for $1 \le i \le 6m+2$, then study the neighbors of z to obtain a contradiction to the assumption that n is even. We distinguish two cases based on whether the two neighbors of z on C are adjacent or not.

Case B1. If the two neighbors of z on C are adjacent, then we can assume that z is adjacent with vertices a_1, a_2 of G. Then z' is adjacent with vertices a_{3m+2}, a_{3m+3} of C. See Fig. 13.

We first show that for $1 \le i \le 3m+1$,

$$|A_i| = |A_{3m+1+i}| = \begin{cases} n+1, & \text{if } i = 3j \text{ or } i = 3j+1 \text{ for } 1 \le j \le m, \\ n, & \text{otherwise.} \end{cases} \tag{B5}$$

Since each vertex of C has degree $3n+1$, we have the following observations for the size of the union of three consecutive A_i's.

(i) For $i = 1, 2, 3m+2, 3m+3$, $|A_{i-1} \cup A_i \cup A_{i+1}| \ge 3n+1$ since a_i can only be adjacent with vertices from $\{z\} \cup A_{i-1} \cup (A_i \setminus \{a_i\}) \cup A_{i+1}$ or $\{z'\} \cup A_{i-1} \cup (A_i \setminus \{a_i\}) \cup A_{i+1}$;

(ii) For other i, $|A_{i-1} \cup A_i \cup A_{i+1}| \ge 3n+2$ since a_i can only be adjacent with vertices from $A_{i-1} \cup (A_i \setminus \{a_i\}) \cup A_{i+1}$.

By these observations and Eq. (B4), we get the following inequalities:

$$\begin{aligned} |A_1| &= \left|\left(\bigcup_{1\le i\le 3m+1} A_i\right) \setminus \left(\bigcup_{1\le k\le m} (A_{3k-1} \cup A_{3k} \cup A_{3k+1})\right)\right| \\ &\le \frac{1}{2}[2(3m+1)n + 4m] - m(3n+2) = n, \end{aligned}$$

and for $1 \le j \le m$,

$$\begin{aligned} |A_{3j-1}| &= \left|\left(\bigcup_{1\le i\le 3m+1} A_i\right) \setminus \left(\bigcup_{j\le k\le m-1} (A_{3k} \cup A_{3k+1} \cup A_{3k+2})\right)\right. \\ &\qquad \left. \setminus (A_{3m} \cup A_{3m+1} \cup A_1) \setminus \left(\bigcup_{1\le k\le j-1} (A_{3k-1} \cup A_{3k} \cup A_{3k+1})\right)\right| \\ &= \left|\left(\bigcup_{1\le i\le 3m+1} A_i\right) \setminus \left(\bigcup_{j\le k\le m-1} (A_{3k} \cup A_{3k+1} \cup A_{3k+2})\right)\right. \\ &\qquad \left. \setminus (A_{3m} \cup A_{3m+1} \cup A_{3m+2}) \setminus \left(\bigcup_{1\le k\le j-1} (A_{3k-1} \cup A_{3k} \cup A_{3k+1})\right)\right| \\ &\le \frac{1}{2}[2(3m+1)n + 4m] - m(3n+2) = n, \\ |A_{3j}| &= \left|\left(\bigcup_{1\le i\le 3m+1} A_i\right) \setminus \left(\bigcup_{j\le k\le m-1} (A_{3k+1} \cup A_{3k+2} \cup A_{3k+3})\right)\right. \\ &\qquad \left. \setminus (A_{3m+1} \cup A_1 \cup A_2) \setminus \left(\bigcup_{1\le k\le j-1} (A_{3k} \cup A_{3k+1} \cup A_{3k+2})\right)\right| \\ &= \left|\left(\bigcup_{1\le i\le 3m+1} A_i\right) \setminus \left(\bigcup_{j\le k\le m-1} (A_{3k+1} \cup A_{3k+2} \cup A_{3k+3})\right)\right. \\ &\qquad \left. \setminus (A_{3m+1} \cup A_{3m+2} \cup A_{3m+3}) \setminus \left(\bigcup_{1\le k\le j-1} (A_{3k} \cup A_{3k+1} \cup A_{3k+2})\right)\right| \\ &\le \frac{1}{2}[2(3m+1)n + 4m] - [(m-1)(3n+2) + (3n+1)] = n+1, \end{aligned}$$

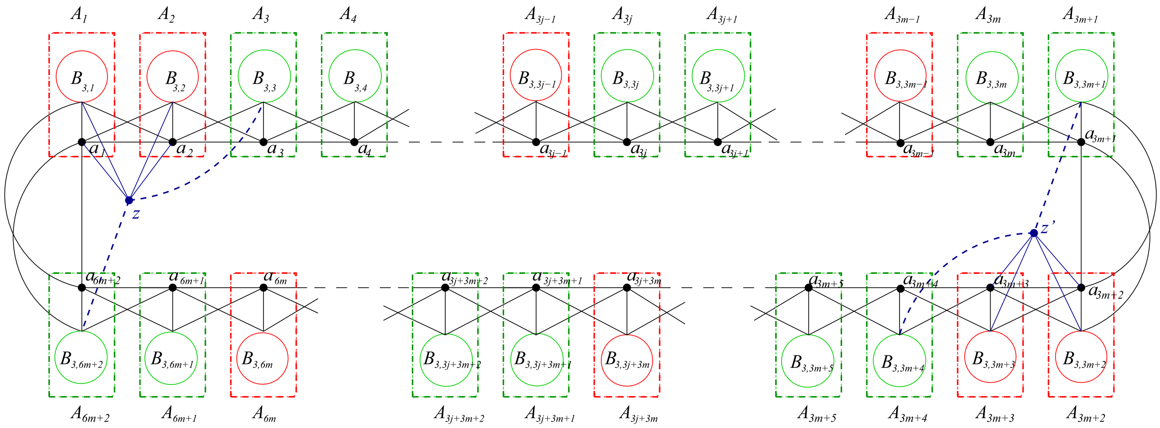


Fig. 13. Illustration for G considered in the proof of Case B1.

$$
\begin{aligned}
|A_{3j+1}| &= \Bigg|\left(\bigcup_{1\le i\le 3m+1} A_i\right) \setminus \left(\bigcup_{j\le k\le m-1} (A_{3k+2}\cup A_{3k+3}\cup A_{3k+4})\right) \\
&\quad \setminus \Big(\bigcup_{1\le k\le j} (A_{3k-2}\cup A_{3k-1}\cup A_{3k})\Big)\Bigg| \\
&\le \frac{1}{2}[2(3m+1)n+4m] - [(m-1)(3n+2)+(3n+1)] = n+1.
\end{aligned}
$$

Adding all the above inequalities, we get $\sum_{i=1}^{3m+1}|A_i| \le (3m+1)n+2m$.

Since $|A_i| = |A_{3m+1+i}|$ for all $1\le i\le 3m+1$, we have $\sum_{i=1}^{6m+2}|A_i| \le 2[(3m+1)n+2m]$. By Eq. (B4), we can see that only the equality holds. Thus, for all the above inequalities on $|A_i|$, only the equalities hold. This proves Eq. (B5).

Since C is a strongly induced cycle of G, it is easy to see that the neighbors of z are contained in $B_{3,6m+2}\cup A_1\cup A_2\cup B_{3,3}$, and each vertex u of A_1 (resp. A_2) can only be adjacent with vertices from $\{z\}\cup A_{6m+2}\cup(A_1\setminus\{u\})\cup A_2$ (resp. $\{z\}\cup A_1\cup(A_2\setminus\{u\})\cup A_3$).

Note that G is $(3n+1)$-regular and that $|A_{6m+2}| = n+1$ and $|A_1| = |A_2| = n$. Then A_1 must be a clique K_n and each vertex u of A_1 must be adjacent with all $2n+2$ vertices of $\{z\}\cup A_{6m+2}\cup A_2$. Similarly, A_2 must be a clique K_n and each vertex v of A_2 must be adjacent with all $2n+2$ vertices of $\{z\}\cup A_1\cup A_3$. Thus, z is adjacent with all $2n$ vertices of $A_1\cup A_2$. It follows that z must be adjacent with $n+1$ vertices from $B_{3,6m+2}\cup B_{3,3}$ since $\deg(z) = 3n+1$.

Now we can show that $n+1$ is even to obtain a contradiction. Assume that $x\in B_{3,6m+2}$ is adjacent with z. Then x can only be adjacent with vertices from $\{z\}\cup A_{6m+1}\cup(A_{6m+2}\setminus\{x\})\cup A_1$. In last paragraph, we already see that x is adjacent with all n vertices of A_1. Moreover, we can show that x is adjacent with all $n+1$ vertices of A_{6m+1} as follows. Each vertex of A_{6m+1} can only be adjacent with vertices from $A_{6m}\cup A_{6m+1}\cup A_{6m+2}$, and we have shown that $|A_{6m}| = n$, $|A_{6m+1}| = |A_{6m+2}| = n+1$. Then A_{6m+1} must be a clique K_{n+1} and each vertex of A_{6m+1} is adjacent with all $(2n+1)$ vertices of $A_{6m}\cup A_{6m+2}$. It implies that $x\in B_{3,6m+2}$ is adjacent with all $n+1$ vertices of A_{6m+1}. Therefore, $x\in B_{3,6m+2}$ is adjacent with all $2n+2$ vertices of $\{z\}\cup A_{6m+1}\cup A_1$. Then x is adjacent with exactly $n-1$ vertices of A_{6m+2} since $\deg(x) = 3n+1$. So, there exists a unique vertex $y\in B_{3,6m+2}\setminus\{x\}$ such that x is adjacent with all $n-1$ vertices of $A_{6m+2}\setminus\{x,y\}$. It is clear that y must be adjacent with z and all vertices of $A_{6m+2}\setminus\{x,y\}$ since $\deg(y) = 3n+1$. Similarly, we can show that if $u\in B_{3,3}$ is adjacent with z, then there exists a unique vertex $v\in B_{3,3}\setminus\{u\}$ such that u and v are not adjacent and both of them are adjacent with z and all vertices of $A_3\setminus\{u,v\}$. It implies that $n+1$, the number of neighbors of z in $B_{3,6m+2}\cup B_{3,3}$, is even. This contradicts the assumption that n is even. Hence, Case B1 of Claim B is proved.

Case B2. The two neighbors of z on C are not adjacent. Then they must be of distance 2 since C is a strongly induced $2(3m+1)$-cycle of G. We may assume that z is adjacent with a_1 and a_3 of C, and so z' is adjacent with a_{3m+2} and a_{3m+4} of C. See Fig. 14.

Since each vertex of C has degree $3n+1$ in G, we have the following observations on the size of the union of three consecutive A_i's.

(i) For $i = 1, 3, 3m+2, 3m+4$, $|A_{i-1}\cup A_i\cup A_{i+1}| \ge 3n+1$, since a_i can only be adjacent with vertices from $\{z\}\cup A_{i-1}\cup(A_i\setminus\{a_i\})\cup A_{i+1}$ or $\{z'\}\cup A_{i-1}\cup(A_i\setminus\{a_i\})\cup A_{i+1}$;

(ii) For other i, $|A_{i-1}\cup A_i\cup A_{i+1}| \ge 3n+2$, since a_i can only be adjacent with vertices from $A_{i-1}\cup(A_i\setminus\{a_i\})\cup A_{i+1}$.

In a way similar to the proof of Case B1, we can get a contradiction by the following steps.

First, we show that for $1\le i\le 3m+1$,

$$
|A_i| = |A_{3m+1+i}| = \begin{cases} n+1, & \text{if } i=1 \text{ or } i=3j \text{ for } 1\le j\le m, \text{ or} \\ & \text{if } i=3j-1 \text{ for } 2\le j\le m, \\ n, & \text{otherwise.} \end{cases}
$$

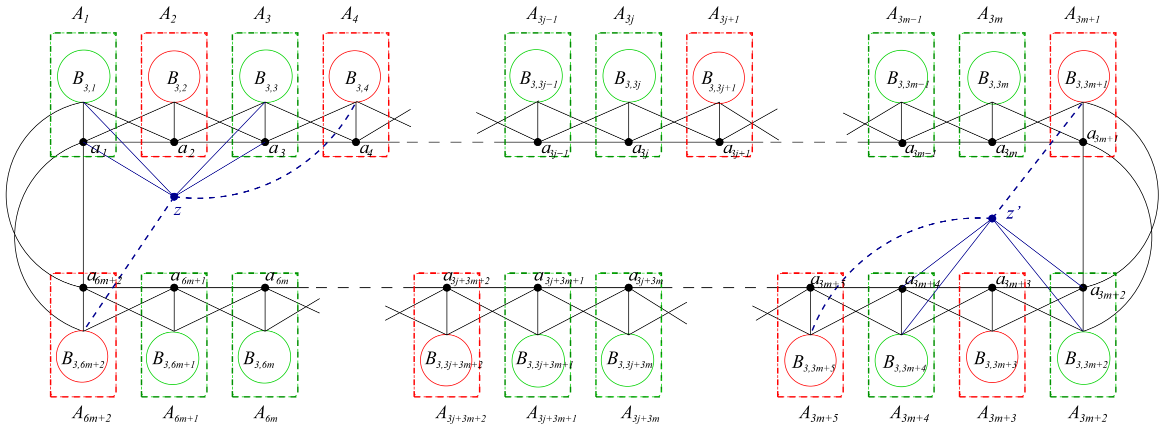


Fig. 14. Illustration for G considered in the proof of Case B2.

Second, the neighbors of z are contained in $B_{3,6m+2} \cup A_1 \cup A_3 \cup B_{3,4}$. We can show that z is adjacent with all $2n+2$ vertices of $A_1 \cup A_3$. Hence, z must be adjacent with $n-1$ vertices from $B_{3,6m+2} \cup B_{3,4}$ since $\deg(z) = 3n+1$.

Finally, we can show that $n-1$, the number of neighbors of z in $B_{3,6m+2} \cup B_{3,4}$, is even to obtain a contradiction to the assumption that n is even.

This proves for Case B2, and so the proof of Claim B is complete. □

4. Open question

For any given positive integers r and k, let $\mathbb{B}(r, k)$ denote the set of bipartite r-regular graphs with pair length k. What is the minimum vertex number of a graph in $\mathbb{B}(r, k)$?

We think this question is challenging although it is easy to answer for the special case when $r \leq 2$ or $k \leq 2$.

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References

[1] S. Al-Addasi, H. Al-Ezeh, Bipartite diametrical graphs of diameter 4 and extreme orders, International Journal of Mathematics and Mathematical Sciences 2008 (2008) 11. doi:10.1155/2008/468583.
[2] A. Berman, A. Kotzig, Cross-cloning and antipodal graphs, Discrete Mathematics 69 (1988) 107–114.
[3] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Elsevier, Science Ltd., 1976.
[4] Z. Che, On k-pairable graphs from trees, Czechoslovak Mathematical Journal 57 (2007) 377–386.
[5] Z. Che, Uniquely pairable graphs, Discrete Mathematics 308 (2008) 6104–6110.
[6] Z. Chen, On k-pairable graphs, Discrete Mathematics 287 (2004) 11–15.
[7] D. Christofides, Pair lengths of product graphs, Discrete Mathematics 306 (2006) 2111–2114.
[8] N. Graham, R.C. Entringer, L.A. Székely, New tricks for old trees: maps and the pigeonhole principle, American Mathematical Monthly 101 (1994) 664–667.
[9] F. Göbel, H.J. Veldman, Even graphs, Journal of Graph Theory 10 (1986) 225–239.
[10] D.A. Holton, J. Sheehan, The Petersen Graph, in: Australian Mathematical Society Lecture Series, Cambridge University Press, 1993.
[11] W. Imrich, S. Klavžar, Product Graphs: Structure and Recognition, in: Wiley-Interscience Series in Discrete Mathematics and Optimization, 2000.
[12] L. Lovasz, M.D. Plummer, Matching Theory, in: North-Holland Mathematics Studies, vol. 121, North-Holland Publishing Co., Amsterdam, 1986, Annals of Discrete Mathematics, vol 29.
[13] R. Nowakowski, I. Rival, Retract rigid Cartesian products of graphs, Discrete Mathematics 70 (1988) 169–184.
[14] G. Sabidussi, Graph multiplication, Mathematische Zeitschrift 72 (1960) 446–457.